Fill Ups, Subjective Problems of Indefinite Integrals

Fill in the Blanks

Q.1. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B \log (9e^{2x} - 4) + C,$ **then A = ..., B = ... and C =.......**
(1990 - 2 Marks)

Ans. $\frac{-3}{2}, \frac{35}{36}$, any real value

Solutions.

$$\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = Ax + B\,ln\,(9e^{2x} - 4) + C$$

$$\Rightarrow \frac{d}{dx}[Ax + B\ln(9e^{2x} - 4) + C] = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}$$

$$\Rightarrow A + \frac{18Be^x}{9e^x - 4e^{-x}} = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}$$

$$\Rightarrow \frac{(9A + 18B)e^x - 4Ae^{-x}}{9e^x - 4e^{-x}} = \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}$$

$$\Rightarrow 9A + 18B = 4; -4A = 6 \Rightarrow A = \frac{-3}{2};$$

$B = \left(4 + \frac{27}{2}\right)\frac{1}{18} = \frac{35}{36};\; C$ Can have any real value.

Subjective Problems

Q. 1. Evaluate $\int \frac{\sin x}{\sin x - \cos x} dx$ **(1978)**

Ans. $\frac{1}{2}\log|\sin x - \cos x| + \frac{x}{2} + C$

Solutions.

$$I=\int\frac{\sin x}{\sin x-\cos x}dx \quad =\frac{1}{2}\int\frac{2\sin x}{\sin x-\cos x}dx$$

$$=\frac{1}{2}\int\frac{\sin x+\cos x+\sin x-\cos x}{\sin x-\cos x}dx$$

$$=\frac{1}{2}\int\frac{\cos x+\sin x}{\sin x-\cos x}dx+\frac{1}{2}\int\frac{\sin x-\cos x}{\sin x-\cos x}dx$$

$$=\frac{1}{2}\log|\sin x-\cos x|+\frac{x}{2}+C$$

Q. 2. Evaluate $\int\frac{x^2dx}{(a+bx)^2}$ **(1979)**

Ans. $\frac{1}{b^3}\left[a+bx-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+C$

Solutions.

Let $I=\int\frac{x^2dx}{(a+bx)^2}$

Let $a+bx=t \Rightarrow x=\left(\frac{t-a}{b}\right) \Rightarrow dx=\frac{dt}{b}$

$$\therefore \quad I=\frac{1}{b^3}\int\frac{(t-a)^2}{t^2}dt \quad =\frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}dt$$

$$=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt \quad =\frac{1}{b^3}\left[t-2a\log|t|-\frac{a^2}{t}\right]+C$$

$$=\frac{1}{b^3}\left[a+bx-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+C$$

Q. 3. Evaluate $\int(e^{\log x}+\sin x)\cos x\,dx$. **(1981 - 2 Marks)**

Ans. $x\sin x+\cos x-\frac{1}{4}\cos 2x+C$

Solutions.

To evaluate $\int (e^{\log x} + \sin x)\cos x\, dx$

$= \int (x + \sin x)\cos x\, dx$ [Using $e^{\log x} = x$]

$= \int x\cos x + \frac{1}{2}\int \sin 2x\, dx$

$= x\sin x - \int \sin x\, dx + \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$

$= x\sin x + \cos x - \frac{1}{4}\cos 2x + C$

Q. 4. Evaluate : $\int \frac{(x-1)e^x}{(x+1)^3} dx$ **(1983 - 2 Marks)**

Ans. $\frac{e^x}{(x+1)^2} + C$ 5. $-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$

Solutions.

$I = \int \frac{(x-1)e^x}{(x+1)^3} dx = \int \frac{(x+1-2)e^x}{(x+1)^3} dx$

$= \int \left[\frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}\right] e^x\, dx = \frac{e^x}{(x+1)^2} + C$

(Using $\int e^x (f(x) + f'(x)dx = e^x f(x))$)

Q. 5. Evaluate the following $\int \frac{dx}{x^2(x^4+1)^{3/4}}$ **(1984 - 2 Marks)**

Ans. $-\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + C$

Solutions.

Let $I = \int \frac{dx}{x^3 x^2 \left(1+\frac{1}{x^4}\right)^{3/4}}$

Put $1+\frac{1}{x^4} = t \Rightarrow \frac{-4}{x^5}dx = dt \Rightarrow \frac{dx}{x^5} = -\frac{dt}{4}$

$$\therefore \quad I = \int \frac{-dt}{4t^{3/4}} = \frac{-1}{4}\left\{\frac{t^{-3/4+1}}{\frac{-3}{4}+1}\right\} + C$$

$$= -t^{1/4} + C = -\left(1+\frac{1}{x^4}\right)^{1/4} + C$$

Q. 6. Evaluate the following $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$ **(1985 - 2½ Marks)**

Ans. $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$

Solutions.

$I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

Put $x = \cos^2\theta \Rightarrow dx = -2\cos\theta\sin\theta\, d\theta$

$$\therefore \quad I = -\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.2\sin\theta\cos\theta\, d\theta$$

$$= -\int \frac{\sin\theta/2}{\cos\theta/2}.2.2\sin(\theta/2)\cos(\theta/2)\cos\theta\, d\theta$$

$$= -2\int (1-\cos\theta)\cos\theta\, d\theta$$

$$= -2\int (\cos\theta - \cos^2\theta)\, d\theta$$

$$= -2\int \left(\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= -2\left[\sin\theta - \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C\right]$$

$$= -2\sqrt{1-x} + \left[\cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x}\right] + C$$

[Using $\sin\theta = \sqrt{1-x}$]

$$= -\ 2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x}\sqrt{1-x} + C$$

Q. 7. Evaluate : $\int \left[\dfrac{(\cos 2x)^{1/2}}{\sin x}\right] dx$ **(1987 - 6 Marks)**

Ans. $\dfrac{1}{\sqrt{2}}\log\left[\dfrac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right] - \log(\cot x + \sqrt{\cot^2 x - 1}) + C$

Solutions.

$$I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx = \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin^2 x} dx = \int \sqrt{\cot^2 x - 1}\ dx$$

Let $\cot x = \sec\theta \Rightarrow -\text{cosec}^2 x\ dx = \sec\theta \tan\theta\ d\theta$

We get, $I = \int \sqrt{\sec^2\theta - 1}.\dfrac{\sec\theta\tan\theta}{-(1+\sec^2\theta)} d\theta$

$$= -\int \frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta} d\theta = -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta} d\theta$$

$$= -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)} d\theta = -\int \frac{(1+\cos^2\theta) - 2\cos^2\theta}{\cos\theta(1+\cos^2\theta)} d\theta$$

$$= -\int \sec\theta d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta} d\theta$$

$$= -\log|\sec\theta + \tan\theta| + 2\int \frac{\cos\theta}{2-\sin^2\theta} d\theta$$

$$= -\log|\sec\theta + \tan\theta| + 2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + C$$

$$= -\log\left|\cot x + \sqrt{\cot^2 x - 1}\right| + \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right| + C$$

Q. 8. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x}) dx$ **(1989 - 3 Marks)**

Ans. $\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+c$

Solutions.

$$I=\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\sqrt{\sin x}}{\sqrt{\cos x}}+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}dx$$

$$=\int\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}dx \;=\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}$$

Put sin x – cos x = t ⇒ (cos x + sin x) dx = dt also (sin x – cos x)^2 = t^2 ⇒ 1 – sin 2x = t$^{2\backslash}$

$\Rightarrow \sin 2x=1-t^2$ $\quad \because I=\sqrt{2}\int\frac{dt}{\sqrt{1-t^2}}$

$$=\sqrt{2}\sin^{-1}t+C \;=\sqrt{2}\sin^{-1}(\sin x-\cos x)+C$$

Q. 9. Find the indefinite integral $\int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{4}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$ **(1992 - 4 Marks)**

Ans.

$$\frac{3}{2}x^{2/3}-\frac{12}{7}x^{7/12}+2x^{1/2}-\frac{12}{5}x^{5/12}+3x^{1/3}-4x^{1/4}+6x^{1/6}-12x^{1/12}$$
$$+12\log|x^{1/2}+1|+6\left[\left\{\frac{(1+x^{1/6})^3}{3}-\frac{3}{2}(1+x^{1/6})^2+3(1+x^{1/6})\right\}\right.$$
$$\ln(1+x^{1/6})-\left\{\frac{(1+x^{1/6})^3}{9}-\frac{3}{4}(1+x^{1/6})^2+3(1+x^{1/6})\right\}$$

Solutions.

$$\text{Let } I=\int\left(\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}+\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}\right)dx$$

$$=\int\frac{1}{\sqrt[3]{x}+\sqrt[4]{x}}dx+\int\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x}+\sqrt{x}}dx$$

$$I=I_1+I_2 \qquad ...(1)$$

where $I_1 = \int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} dx$

Let $x = y^{12}$ so that $dx = 12\, y^{11}\, dy$

$$\therefore \quad I_1 = \int \frac{12y^{11}}{y^4 + y^3} dy = 12\int \frac{y^8}{1+y} dy$$

$$= 12\int \left(y^7 - y^6 + y^5 - y^4 + y^3 - y^2 + y - 1 + \frac{1}{y+1} \right) dy$$

$$= 12\left[\frac{y^8}{8} - \frac{y^7}{7} + \frac{y^6}{6} - \frac{y^5}{5} + \frac{y^4}{4} - \frac{y^3}{3} + \frac{y^2}{2} - y + \log|y+1| \right] + C_1$$

$$= \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3}$$

$$-4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12\log|x^{1/12} + 1| + C_1 \quad \text{...(2)}$$

$$I_2 = \int \frac{\ln(1 + (x)^{1/6})}{(x)^{1/3} + (x)^{1/2}} dx$$

Let $x = z^6$ so that $dx = 6z^5\, dz$

$$= \int \frac{\ln(1+z)}{z^2 + z^3} . 6z^5 dz = \int \frac{6z^3 \ln(z+1)}{z+1} dz$$

Put $z + 1 = t \quad \Rightarrow \quad dz = dt$

$$\therefore \quad I_2 = \int \frac{6(t-1)^3 \ln t}{t} dt = 6\int (t^2 - 3t + 3 - \frac{1}{t}) \ln t\, dt$$

$$= 6\left[\int (t^2 - 3t + 3) \ln t\, dt - \int \frac{1}{t} \ln t\, dt \right]$$

$$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \int \left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right). \quad \frac{1}{t} dt - \frac{(\ln t)^2}{2} \right]$$

$$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \int \left(\frac{t^2}{3} - \frac{3}{2}t + 3 \right) dt \quad - \frac{(\ln t)^2}{2} \right]$$

$$= 6\left[\left(\frac{t^3}{3} - \frac{3t^2}{2} + 3t \right) \ln t - \left(\frac{t^3}{9} - \frac{3t^2}{4} + 3t \right) - \frac{(\ln t)^2}{2} \right] + C_2$$

$$= 6\left[\left\{\frac{(1+x^{1/6})^3}{3} - \frac{3}{2}(1+x^{1/6})^2 + 3(1+x^{1/6})\right\}\right.$$

$$\left. \ln(1+x^{1/6}) - \left\{\frac{(1+x^{1/6})^3}{9} - \frac{3}{4}(1+x^{1/6})^2 + 3(1+x^{1/6})\right\} + \frac{[\ln(1+x^{1/6})^2]}{2}\right] + C_2$$ ...(3)

Thus we get the value of I on substituting the values of I_1 and I2 from (2) and (3) in equation (1).

Q. 10. Find the indefinite integral $\int \cos 2\theta \ln\left(\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\right) d\theta$ **(1994 - 5 Marks)**

Ans. $\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \frac{1}{2}\ln\sec 2\theta + C$

Solutions.

Let $I = \int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) d\theta$

Now we observe that

$$\frac{d}{d\theta}\left\{\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right\}$$

$$= \frac{d}{d\theta}[\ln(\cos\theta+\sin\theta) - \ln(\cos\theta-\sin\theta)]$$

$$= \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} - \frac{-\cos\theta-\sin\theta}{\cos\theta-\sin\theta}$$

$$= \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} + \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$$

$$= \frac{\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta} = \frac{2}{\cos 2\theta}$$

∴ Integrating I with respect to θ, by parts we get

$$I=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int\frac{\sin 2\theta}{2}\cdot\frac{2}{\cos 2\theta}d\theta$$

$$=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\int\tan 2\theta\, d\theta$$

$$=\frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)-\frac{1}{2}\ln\sec 2\theta+C$$

Q. 11. Evaluate $\int\frac{(x+1)}{x(1+xe^x)^2}dx.$ **(1996 - 2 Marks)**

Ans. $\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+C$

Solutions.

$$I=\int\frac{(x+1)}{x(1+xe^x)^2}dx=\int\frac{e^x(x+1)}{xe^x(1+xe^x)^2}dx$$

Put $1+xe^x=t\Rightarrow(xe^x+e^x)\,dx=dt$

$$\therefore\quad I=\int\frac{dt}{(t-1)t^2}=\int\left(\frac{1}{1-t}+\frac{1}{t}+\frac{1}{t^2}\right)dt$$

$$=-\log|1-t|+\log|t|-\frac{1}{t}+C$$

$$=\log\left|\frac{t}{1-t}\right|-\frac{1}{t}+C=\log\left|\frac{1+xe^x}{-xe^x}\right|-\frac{1}{1+xe^x}+C$$

$$=\log\left(\frac{1+xe^x}{xe^x}\right)-\frac{1}{1+xe^x}+C$$

Q. 12. Integrate $\int\frac{x^3+3x+2}{\left(x^2+1\right)^2(x+1)}dx.$ **(1999 - 5 Marks)**

Ans. $-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{3}{2}\tan^{-1}x+\frac{x}{1+x^2}+C$ $\left.+\frac{(\ln(1+x^{1/6}))^2}{2}\right]+C$

Solutions.

$$I=\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}dx.$$

$$\frac{x^3+3x+2}{(x^2+1)^2(x+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

Comparing and solving, we get,

$$A=\frac{1}{2},B=\frac{1}{2},C=\frac{1}{2},D=0,E=2.$$

$$\therefore\quad I=-\frac{1}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{x+1}{x^2+1}dx+2\int\frac{dx}{(x^2+1)^2}$$

$$=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+2I_1+C$$

where $I_1=\int\frac{dx}{(x^2+1)^2}$, putting $x=\tan\theta$,

$$I_1=\int\frac{\sec^2\theta}{\sec^4\theta}d\theta=\int(\cos^2\theta)\,d\theta=\frac{1}{2}\int(1+\cos 2\theta)d\theta$$

$$=\frac{1}{2}(\theta+\frac{1}{2}\sin 2\theta)=\frac{1}{2}\tan^{-1}x+\frac{1}{4}.\frac{2x}{1+x^2}$$

$$\therefore\quad I=-\frac{1}{2}\log|x+1|+\frac{1}{4}\log(x^2+1)+\frac{3}{2}\tan^{-1}x\quad+\frac{x}{1+x^2}+C$$

Where C is constant of integration.

Q. 13. Evaluate $\int\sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$. **(2001 - 5 Marks)**

Ans. $(x+1)\tan^{-1}\left(\frac{2x+2}{3}\right)-\frac{3}{4}\log(4x^2+8x+13)+C$

Solutions.

$$I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$$

$$= \int \sin^{-1}\left[\frac{x+1}{\sqrt{x^2+2x+\frac{13}{4}}}\right]dx$$

$$= \int \sin^{-1}\left[\frac{x+1}{\sqrt{(x+1)^2)+(3/2)^2}}\right]dx$$

Put $x+1 = 3/2\tan\theta,\ dx = \frac{3}{2}\sec^2 d\theta$

$$\therefore \quad I = \int \sin^{-1}\left[\frac{\left(\frac{3}{2}\tan\theta\right)}{\sqrt{\frac{9}{4}\tan^2\theta+\frac{9}{4}}}\right]\frac{3}{2}\sec^2\theta\, d\theta$$

$$= \frac{3}{2}\int \sin^{-1}\left[\frac{\sin\theta}{\cos\theta}\frac{\cos\theta}{1}\right]\sec^2\theta d\theta$$

$$= \frac{3}{2}\int \theta\sec^2\theta\, d\theta = \frac{3}{2}\left[\theta\tan\theta - \int \tan\theta\, d\theta\right]$$

$$= \frac{3}{2}[\theta\tan\theta - \log|\sec\theta|] + C$$

$$I = \frac{3}{2}\left[\frac{2}{3}(x+1)\tan^{-1}\right]\left[\frac{2}{3}(x+1)\right] - \log\sqrt{1+\frac{4}{9}(x+1)^2]} + C$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(9+4x^2+8x+4) \quad + \frac{3}{4}\log 9 + C$$

$$= (x+1)\tan^{-1}\left(\frac{2x+2}{3}\right) - \frac{3}{4}\log(4x^2+8x+13) + C$$

Q. 14. For any natural number m, evaluate $\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{l/m}dx$, $x > 0$. **(2002 - 5 Marks)**

Ans. $\frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1}+C$

Solutions.

$$=\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}dx$$

$$=\int (x^{3m}+x^{2m}+x^m)\left[\frac{2x^{3m}+3x^{2m}+6x^m}{x^m}\right]^{1/m}dx$$

$$=\int\left(\frac{x^{3m}+x^{2m}+x^m}{x}\right)(2x^{3m}+3x^{2m}+6x^m)^{1/m}dx$$

Put $2x^{3m}+3x^{2m}+6x^m=y$

$$\therefore \quad I=\frac{1}{6m}\int y^{1/m}dy=\frac{1}{6m}\left(\frac{y^{1/m+1}}{1/m+1}\right)+C$$

$$=\frac{1}{6}\left(\frac{y^{\frac{m+1}{m}}}{m+1}\right)+C$$

$$=\frac{1}{6}\frac{(2x^{3m}+3x^{2m}+6x^m)^{\frac{m+1}{m}}}{m+1}+C$$